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Principle of Equivalence and Electromagnetism

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We show that the local expression of the electrostatic potential of a point charge suggested from the equivalence principle is different of the one resulting from the global consideration in the Schwarzschild space-time.

1. INTRODUCTION

The origins of general relativity are mainly based on two general principles: the principle of general covariance and the principle of equivalence. The latter states that in a small region of space-time gravitational forces are indistinguishable from inertial forces. Thus it is impossible, on the basis of purely local experiments, to distinguish between these two forces (Einstein, 1916; Tonnelat, 1971; Ehlers, 1973).

In special relativity, it is possible to choose an inertial coordinate system such that a free particle should have no acceleration. For the special coordinates used, the metric tensor of Minkowski space-time has the following form:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{1}$$

Except for the gravitational law, one postulates the standard expressions of the laws of physics in this space. In particular Maxwell equations may be written in Minkowski coordinates as

$$\partial_\alpha F^{\alpha\beta} = J^\beta \qquad \text{and} \qquad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{2}$$

In general relativity, at every space-time point one can consider a region small enough so that the gravitational field is sensibly constant

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throughout it. There always exists a coordinate system covering this region in which the metric can be transformed in the form (1). That is the locally inertial coordinate system such that the laws of physics, by virtue of the principle of equivalence, have the same form as in an inertial coordinate system in Minkowski space. But this principle is not sufficient for writing the equations of laws of physics in general relativity. In fact, the equations are postulated by adopting a principle of minimal gravitational coupling. Thus the generalization of Maxwell equations (2) in curved space are the following equations:

$$\nabla_\alpha F^{\alpha\beta} = J^\beta \qquad \text{and} \qquad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0 \tag{3}$$

Now what is the problem? In the manner described above, in this infinitely small region of space-time, a homogeneous gravitational field is equivalent to an accelerated coordinate system in Minkowski space in the sense that the form of the equations of laws of physics are the same in both cases. However, one should not infer that local measurements will be identical because the solution of the equations in a locally homogeneous gravitational field are the restriction of the global solution determined in the global space-time taking into account the boundary conditions at the infinity. The purpose of this paper is to show, in a particular situation, that this is true. Our approach confirms, by an alternative point of view, the recent result of Smith and Will (1980).

We consider a point charge at rest in the Schwarzschild space-time. The influence of the electromagnetic field on the metric is assumed to be negligible. In a small neighborhood of the position of the charge, the gravitational field is a constant, static, and homogeneous gravitational field. In Section 2, we use in our small region the electrostatic potential calculated from a uniformly accelerated coordinate system in Minkowski space. In Section 3, we deduce the electrostatic potential in our small region from the known electrostatic potential in Schwarzschild metric. In Section 4, these results may help us investigate the questions raised in the above paragraphs.

2. POTENTIAL SUGGESTED BY THE PRINCIPLE OF EQUIVALENCE

Firstly, we are going to verify that in a small neighborhood of the point in Schwarzschild space-time the gravitational field is homogeneous. In fact one can take a line because the metric is static. We start from the

Schwarzschild metric in standard coordinates:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{1-2m/r} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 \tag{4}$$

and we consider a line (C) located at $r = R$, $\theta = \pi/2$, and $\varphi = 0$. We introduce a coordinate system (y^0, y^i) which is related to the standard Schwarzschild coordinates by the transformation

$$t = \frac{1}{\Delta} y^0$$

$$r = R + \Delta y^1 + \frac{m}{2R^2}(y^1)^2 + \frac{\Delta^2}{2R}\left[(y^2)^2 + (y^3)^2\right]$$

$$\theta = \frac{\pi}{2} + \frac{1}{R} y^2 - \frac{\Delta}{R^2} y^1 y^2$$

$$\varphi = \frac{1}{R} y^3 - \frac{\Delta}{R^2} y^1 y^3 \qquad \text{with} \qquad \Delta = \left(1 - \frac{2m}{R}\right)^{1/2} \tag{5}$$

so that, in a small neighborhood of the line (C), the metric (4) takes the form

$$ds^2 = (1 + 2gy^1)(dy^0)^2 - (dy^1)^2 - (dy^2)^2 - (dy^3)^2 \qquad \text{with} \qquad g = \frac{m}{R^2\Delta} \tag{6}$$

up to the second-order corrections in the coordinates y^i.

The metric associated with a constant, static, homogeneous gravitational field is well known. It is convenient to use the following form:

$$ds^2 = (1 + g\xi^1)^2 (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2 \tag{7}$$

Comparing (7) with (6), we see that the two metrics coincide in the small neighborhood of the line (C), if we neglect the quadractic terms in ξ^i. Thus Schwarzschild metric (4) allows us to define a homogeneous gravitational field described locally by the metric (6) and characterized by the acceleration:

$$g^i = \frac{m}{R^2\Delta} \delta^i_1 \tag{8}$$

It is known that the coordinate system (ξ^0, ξ^i) is related to the inertial coordinate system by the formula

$$
\begin{aligned}
x^0 &= \left(\frac{1}{g} + \xi^1\right) \sinh(g\xi^0) \\
x^1 &= \left(\frac{1}{g} + \xi^1\right) \cosh(g\xi^0) \\
x^2 &= \xi^2 \\
x^3 &= \xi^3
\end{aligned} \tag{9}
$$

but the coordinate system (ξ^0, ξ^i) covers only a part of Minkowski space defined by $x^1 + x^0 > 0$ and $x^1 - x^0 < 0$.

We are interested in determining the electrostatic potential in the small neighborhood described with the metric (6) with the help of the principle of equivalence. The electromagnetic field generated by a uniformly accelerated point charge q has been a subject of considerable investigation. But as clearly noted by Boulware (1980), in the region of Minkowski space covered by the coordinates (ξ^0, ξ^i) there is no problem. Using the transformations of coordinate systems (9), the electromagnetic field generated by a uniformly accelerated point charge is expressed in coordinates (ξ^0, ξ^i) (Rohrlich, 1963). The electrostatic potential coincides with the one found by Whittaker (1927) corresponding to a point charge at rest in the homogeneous gravitational field. By restriction to a small neighborhood of the line $\xi^i = 0$, we obtain the following potential:

$$
V_W = q \frac{1 + (1/2) g \xi^1}{\left[(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2\right]^{1/2}} \tag{10}
$$

According to the principle of equivalence, the expression (10) would be the electrostatic potential in the neighborhood of the point charge in the Schwarzschild space-time.

3. POTENTIAL FROM SCHWARZSCHILD SPACE-TIME ANALYSIS

In Schwarzschild space-time, the electrostatic potential which has as its source a point charge has been determined by Linet (1976) with the help of the determination in closed form by Copson (1928) of the elementary solution in Hadamard's sense of the equation of potential. The electrostatic potential of a point test charge q held at rest at the point $r=R, \theta=\pi/2$, and $\varphi=0$ has the following expressions:

$$V=q\frac{(r-m)(R-m)-m^2\sin\theta\cos\varphi}{Rr\left[(r-m)^2+(R-m)^2-m^2-2(r-m)\times(R-m)\sin\theta\cos\varphi+m^2\sin^2\theta\cos^2\varphi\right]^{1/2}}+q\frac{m}{Rr} \qquad (11)$$

The first term, which we note V_C, is the potential of Copson (1928) in standard coordinates and the second term must be added in order to satisfy the boundary conditions at infinity: V has the asymptotic form q/r for $r \to \infty$.

In the small neighborhood of the line (C) we transform (11) into the coordinate system (y^0, y^i) introduced by the formula (5). Then, the electrostatic potential (11) has the following expression:

$$V=q\frac{1+(1/2)gy^1}{\left[(y^1)^2+(y^2)^2+(y^3)^2\right]^{1/2}}-q\frac{m}{R^3}y^1 \qquad (12)$$

We note that the first term in (12) comes from the potential V_C. Now, we consider V given by (12) as the local electrostatic potential of a point charge at rest in a constant, static, homogeneous gravitational field.

4. DISCUSSION AND CONSEQUENCES

Comparing (10) with (12), we see that the two methods described above do not give the same electrostatic potential within a sufficiently small neighborhood of the point charge. We remark that V_W and V_C coincide but in (12) a supplementary term

$$-q\frac{m}{R^3}y^1 \qquad (13)$$

exists and it is regular at the position of the charge. Clearly the potentials (10) and (12) satisfy locally the same equation in our small region. But we recall that (13) arises from the second term in (11) added in order to satisfy the boundary conditions at infinity in the global space-time. Consequently, the principle of equivalence is not valid.

The important thing to notice is that the first term in (12) is the self-field of the point charge with a similar interpretation as Minkowski space. Thus, all effects giving a violation of the principle of equivalence come from (13). For example, we are now in a position to calculate the acceleration due to gravity on a point charge. We have already noted the acceleration (8) which is valid for a massive particle. For a charged particle, we have also the Lorentz force. Beside the self-force already mentioned and that we have not to consider, a force arising from the electric field induced by the potential (13) has the following expression:

$$f^i = q^2 \frac{m}{R^3} \delta^i_1 \tag{14}$$

We rediscover in the coordinate system (y^0, y^i) the recent result of Smith and Will (1980). We remark that the authors use a local freely falling frame. Their method is completely different because (14) appears as the consequence of the procedure of renormalization in curved space.

The method we have adopted here can be extended, *mutatis mutandis*, in some static space-times where the electrostatic potential is also known in closed form: Brans–Dicke (Linet and Teyssandier, 1979) and Reissner–Nordström (Léauté and Linet, 1976). On the other hand, the problem of electric or magnetic multipoles is now under consideration.

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